

Time-resolved density correlations as probe of squeezing in toroidal Bose-Einstein condensates

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Abstract. I study the evolution of mean field and linear quantum fluctuations in a toroidal Bose-Einstein condensate, whose interaction strength is quenched from a finite (repulsive) value to zero. The azimuthal *equal-time* density-density correlation function is calculated and shows temporal oscillations with twice the (final) excitation frequencies after the transition. These oscillations are a direct consequence of positive and negative frequency mixing during non-adiabatic evolution. I will argue that a *time-resolved* measurement of the *equal-time* density correlator might be used to calculate the moduli of the Bogoliubov coefficients and thus the amount of squeezing imposed on a mode, i.e., the number of atoms excited out of the condensate.

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1. Introduction

Bose-Einstein condensates in toroidal traps provide an interesting opportunity to study superfluidity and other quantum effects in backgrounds with nontrivial topology [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. These geometries can be created (*inter alia*) by shining a blue-detuned laser onto a harmonically trapped condensate thus creating an effectively repulsive Gaussian core. The energy minimum is shifted away from the centre of the trap to a finite radius r_0 typically of order $100\,\mu\text{m}$ [18, 19, 20]. In present experiments [18, 19, 20], radial trapping is relatively weak so that azimuthal and radial degrees of freedom must be considered in general. Using different trapping techniques, e.g., higher Laguerre-Gaussian beams as proposed in [21], tighter radial confinement could be achieved restricting motion in that direction to the trap ground state and thus making the system effectively one-dimensional.

In the azimuthal direction, the condensate is usually (almost) homogeneous and obeys periodic boundary conditions permitting stationary solutions with persistent current, i.e., non-zero phase gradient, which could be excited through stirring with a laser [22, 23] or orbital angular momentum transfer from a Laguerre-Gaussian beam [18, 19, 24]. By introducing an azimuthal position dependent potential, i.e., putting some small obstacles into the torus, inhomogeneous flow profiles can be generated. In regions with higher potential, the condensate density will be lower and the velocity higher possibly leading to a violation of the Landau criterion and thus an instability if the local speed of sound is smaller than the flow velocity. For sufficiently high potentials, the barrier can be overcome only through tunneling – a realization of the boson Josephson junction, where superfluids with different phases are on either side of an (impenetrable) tunnel barrier [2, 3, 6, 7, 8, 9, 25]

Local violation of the Landau criterion can also be understood in the context of cosmic analogs. As pointed out by Unruh [26], sound waves in irrotational fluids obey exactly the same evolution equations as massless scalar fields in a certain space-time metric, where the effective curvature is generated by the fluid flow, see [27] for review. The transition from sub- to supersonic flow and *vice versa* can be interpreted as sonic horizons. It becomes thus, in principle, possible to study some aspects of cosmic quantum effects in the laboratory, e.g., analogs of Hawking radiation [41, 43, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], or the freezing and amplification of (quantum) fluctuations in expanding spacetimes [42, 43, 44, 27, 46, 45, 33, 47, 48, 49, 50, 51, 52, 53], where the latter could be achieved by varying confinement, thus making the condensate expand [46, 47, 48, 55, 56], or by changing the two-body interaction strength near a Feshbach resonance [59], i.e., reducing the speed of sound [33, 49, 50, 45, 52, 48, 53, 54].

In this Article, I will concentrate on the latter case and consider the evolution of a Bose-Einstein condensate in an isotropic toroidal trap when decreasing the (repulsive) two-body interactions. Initially, radial confinement shall be relatively weak, i.e., of the same order as the chemical potential, such that more than one radial mode is occupied and the system is two-dimensional. An adiabatic reduction of the coupling strength would lead to a lower chemical potential, so that the condensate would eventually become effectively one-dimensional. For rapid variations of the interactions, however, the condensate – classical mean field as well as (linear) fluctuations – might not be able to follow these changes; the system would not stay in its ground state throughout the evolution. The classical order parameter will be homogeneous in the azimuthal direction due to isotropy of the trap, such that any mean field excitations can only

be in the radial direction, e.g., as radial breathing motions. Hence, any azimuthal excitations must originate from the linear (quantum and thermal) fluctuations such that the symmetry of the considered trap allows for an unambiguous discrimination between (classical) mean field and linear quantum fluctuations.

The squeezing of quantum and thermal fluctuations during non-adiabatic evolution is a very generic phenomenon occurring in many different physical setting such that toroidal Bose-Einstein condensates might serve as quantum simulator, e.g., for aspects of cosmic inflation [42, 44]. During inflation, the size of the universe rapidly increased by a huge factor thus stretching the wavelength of any excitation mode. At some point, the quantum fluctuations could not follow the rapid expansion any more; they froze and got amplified, an imprint of which is observable as small anisotropies in the cosmic microwave background – similar to the small density ripples in Bose-Einstein condensates.

Although in both cases quantum fluctuations get squeezed during non-adiabatic evolution, there exist also a few differences between (massless) quantum fields in expanding space-times and Bose-Einstein condensates with decreasing interaction strength: Most notably, the excitation frequencies go to zero in the former case such that the quantum fluctuations truly freeze, whereas the excitation energies are always non-vanishing in the latter case and are generally determined by the confinement of the condensate, e.g., the torus radius. Since, for non-zero quasi-particle frequency, the two-point function of a squeezed state oscillates with twice the excitation frequency, such oscillations in the azimuthal density correlations would provide a signature of the coherent quasi-particle pairs created during the non-adiabatic evolution.

To exemplify this, let me consider the squeezed harmonic oscillator: Canonical position, $\hat{x} = 1/\sqrt{2\omega}(\hat{a} + \hat{a}^\dagger)$, and momentum, $\hat{p} = -i\sqrt{\omega/2}(\hat{a} - \hat{a}^\dagger)$, can be expressed through raising and lowering operators, \hat{a}^\dagger and \hat{a} , whose time-dependences are just given by oscillating phase factors, e.g., $\hat{a}(t) = e^{-i\omega t}\hat{a}(0)$. If the oscillator is in a squeezed state, the vacuum is not defined with respect to \hat{a} but rather with respect to a different operator \hat{b} , i.e., $\hat{b}|0\rangle = 0$, which can be linked to the operators $\hat{a}(0)$ through a Bogoliubov transformation, $\hat{a} = \alpha\hat{b} + \beta^*\hat{b}^\dagger$. Obviously, the variances of position and momentum now become time-dependent, e.g., $\langle\hat{x}^2\rangle \propto |\alpha|^2 + |\beta|^2 + 2|\alpha\beta|\cos(2\omega t)$. Thus, a *time-resolved* measurement of $\langle\hat{x}^2\rangle(t)$ would yield the coefficients from which the moduli of the Bogoliubov coefficients, $|\alpha|$ and $|\beta|$, can be easily inferred and thus the amount of squeezing, i.e., $|\beta|^2$, can be quantified. Similarly, the time-dependence of the azimuthal *equal-time* density correlations might serve as probe for squeezing in toroidal Bose-Einstein condensates. Note that this is different from the dynamical structure function, which is the Fourier transformed of the two-point function at *different* times.

This Article is organized as follows: In the next section, I review the field equations and their linearization, before introducing quasi-particles in Sec. 3. Density-density correlations as observable will be discussed there too with particular emphasis on the time-dependence of the two-point function due to squeezing of the density fluctuations. The employed trap geometry is briefly review in Sec. 4 before I consider two explicit examples in Sec. 5. For realistic trap parameters, the time-dependent density-density correlations are calculated for a sudden quench as well as a smooth tanh-shaped transition of the interaction strength from a finite value to zero. The results will be summarized and discussed in Sec. 6.

2. Field equations

Trapped Bose-Einstein condensate can be described by the interacting Schrödinger field equation (in units where $\hbar = 1$) [60, 61]

$$i\partial_t \hat{\Psi} = \left[-\frac{\nabla^2}{2m} + V(\mathbf{r}) + U(t)\hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi}, \quad (1)$$

with external potential $V(\mathbf{r})$ and two-body interaction strength $U(t)$. Well below transition temperature, most atoms condense in the lowest state, which acquires a macroscopic occupation number. It is therefore convenient to split the field operator $\hat{\Psi}$ into a macroscopic condensate part, which can be treated classically, and small quantum fluctuations [62]

$$\hat{\Psi} = \left(\Psi + \hat{\chi} + \hat{\zeta} \right) \frac{\hat{A}}{\sqrt{\hat{N}}}, \quad (2)$$

where $\hat{N} = \hat{A}^\dagger \hat{A}$ counts the total atom number. The atomic operator \hat{A} commutes with linear and higher-order quantum fluctuations, so that particle number is always conserved. In expansion (2), the order parameter $\Psi = \mathcal{O}(\sqrt{N})$ describes the condensed atoms, $\hat{\chi} = \mathcal{O}(N^0)$ are linear quantum fluctuations, and $\hat{\zeta} \ll \mathcal{O}(N^0)$ higher orders.

Setting formally $U = \mathcal{O}(1/N)$, I can expand the field equation (1) in powers of N and obtain the Gross-Pitaevskii equation for the order parameter Ψ [63]

$$i\partial_t \Psi = \left(-\frac{\nabla^2}{2m} + V(\mathbf{r}) + U(t)|\Psi|^2 \right) \Psi. \quad (3)$$

For the linear fluctuations $\hat{\chi}$ follows the Bogoliubov-de Gennes equation [64]

$$i\partial_t \hat{\chi} = \left(-\frac{\nabla^2}{2m} + V(\mathbf{r}) + 2U(t)|\Psi|^2 \right) \hat{\chi} + U(t)\Psi^2 \hat{\chi}^\dagger, \quad (4)$$

which contains a coupling between $\hat{\chi}$ and its adjoint $\hat{\chi}^\dagger$. The equation of motion for higher-order fluctuations $\hat{\zeta}$ reads

$$\begin{aligned} i\partial_t \hat{\zeta} = & \left(-\frac{\nabla^2}{2m} + V(\mathbf{r}) + 2U(t)|\Psi|^2 \right) \hat{\zeta} + U(t)\Psi^2 \hat{\zeta}^\dagger \\ & + U(t) (2\Psi \hat{\chi}^\dagger \hat{\chi} + \Psi^* \hat{\chi}^2 + \hat{\chi}^\dagger \hat{\chi}^2) + \mathcal{O}[U(t)\hat{\zeta}] \end{aligned} \quad (5)$$

with terms quadratic and cubic in $\hat{\chi}$ in the second line acting as source for $\hat{\zeta}$. These higher orders $\hat{\zeta}$ need to be small for the mean-field expansion (2) to be valid. Throughout the rest of this Article, I will not regard Eq. (5) any further and concentrate on Eqs. (3) for the background and (4) for the linear quantum fluctuations.

Alternatively to the Gross-Pitaevskii equation (3), the evolution of the condensate can be described through Bernoulli and continuity equations for mean field density, $\varrho_0 = |\Psi|^2$, and phase, $S_0 = \arg \Psi$. The corresponding linear quantum fluctuations are

$$\delta\hat{\varrho} = \Psi^* \hat{\chi} + \Psi \hat{\chi}^\dagger, \quad \delta\hat{S} = (1/2i\varrho_0)(\Psi^* \hat{\chi} - \Psi \hat{\chi}^\dagger) \quad (6)$$

and kinematics of the phase fluctuations $\delta\hat{S}$ would, in the low energy limit, obey the same evolution equations as a scalar field in a certain curved space-time, see, e.g., [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 45, 46, 47, 48, 49, 50, 51, 52, 53] – tools and techniques from general relativity might be applied. However, in view of the factor $1/\varrho_0$ appearing in the expression for $\delta\hat{S}$, linear phase fluctuations are only well-defined for sufficiently large background densities ϱ_0 – which must evidently fail near the boundary of a trapped condensate as the smallness of $\delta\hat{S}$ cannot be guaranteed.

Nonetheless, it is still possible to introduce self-adjoint operators

$$\hat{\chi}_+ = \frac{1}{\sqrt{2}} (\hat{\chi} + \hat{\chi}^\dagger), \quad \hat{\chi}_- = \frac{1}{\sqrt{2}i} (\hat{\chi} - \hat{\chi}^\dagger) \quad (7)$$

similar to those in Eq. (6) but without prefactors $\sqrt{\varrho_0^{\pm 1}}$ since these would eventually void linearization of phase and density in regions with small (background) density. The evolution equations for $\hat{\chi}_\pm$ are most conveniently written by grouping the linear fluctuation operators into a two-vector (see also [30])

$$\partial_t \begin{pmatrix} \hat{\chi}_+ \\ \hat{\chi}_- \end{pmatrix} = \begin{pmatrix} \mathcal{C} & \mathcal{A} \\ -\mathcal{B} & -\mathcal{C} \end{pmatrix} \begin{pmatrix} \hat{\chi}_+ \\ \hat{\chi}_- \end{pmatrix} = \underline{\underline{\mathcal{D}}} \begin{pmatrix} \hat{\chi}_+ \\ \hat{\chi}_- \end{pmatrix}, \quad (8)$$

where the coefficients of the 2×2 matrix $\underline{\underline{\mathcal{D}}}$

$$\begin{aligned} \mathcal{A}(\mathbf{r}, t) &= -\frac{\nabla^2}{2m} + V(\mathbf{r}, t) + 2U(t)|\Psi|^2 - U(t)\Re\Psi^2, \\ \mathcal{B}(\mathbf{r}, t) &= -\frac{\nabla^2}{2m} + V(\mathbf{r}, t) + 2U(t)|\Psi|^2 + U(t)\Re\Psi^2, \\ \mathcal{C}(\mathbf{r}, t) &= U(t)\Im\Psi^2 \end{aligned} \quad (9)$$

are self-adjoint Hilbert-space operators [acting on $L^2(\mathbb{R}^D)$] depending on trap potential $V(\mathbf{r}, t)$, interaction strength $U(t)$, as well as classical background $\Psi(\mathbf{r}, t)$. In (9), $\Re\Psi^2$ and $\Im\Psi^2$ are real and imaginary parts of Ψ^2 , respectively.

3. Quasi-particles and observables

3.1. Quasi-particles

In order to allow for an unambiguous definition of quasi-particles and thus also the vacuum state, I will assume that the condensate is initially in a stationary state, i.e., that the order parameter performs trivial oscillations, $\Psi(t) = e^{-i\mu t}\Psi(0)$ with chemical potential μ . Quasi-particles can then, in principle, be defined as eigenmodes of Eqs. (8), i.e., by diagonalizing $\underline{\underline{\mathcal{D}}}$, see also [65], though one needs to be careful because of the explicit appearance of real and imaginary parts of Ψ^2 in the field equations: Even if the order parameter performs only trivial oscillations, $\Psi(t) = e^{-i\mu t}\Psi(0)$, the coefficients of Eq. (8) would be time-dependent and an instantaneous diagonalization of $\underline{\underline{\mathcal{D}}}$ would not yield the proper fluctuation eigenmodes. It is therefore necessary to absorb the time-dependence of the initial phase in the definition of the quantum fluctuations (which is implicitly done in the fluid-dynamic description) before diagonalizing their evolution equations, see [30]. (Note, however, that the order parameter might retain a space-dependent phase.)

Following Ref. [30], the linear field operators $\hat{\chi}_\pm$ can be expanded in terms of initial quasi-particle solutions (this expansion can also be derived the other way round starting from the eigenfunctions of $\underline{\underline{\mathcal{D}}}$ [65])

$$\begin{aligned} \hat{\chi}_+(\mathbf{r}, t) &= \mathbf{u}_\lambda(\mathbf{r}, t)\hat{a}_\lambda + \mathbf{u}_\lambda^*(\mathbf{r}, t)\hat{a}_\lambda^\dagger, \\ \hat{\chi}_-(\mathbf{r}, t) &= \mathbf{v}_\lambda(\mathbf{r}, t)\hat{a}_\lambda + \mathbf{v}_\lambda^*(\mathbf{r}, t)\hat{a}_\lambda^\dagger, \end{aligned} \quad (10)$$

where each eigenvalue pair $\pm\lambda$ of $\underline{\underline{\mathcal{D}}}$ is summed only once. The entire space-time dependence of $\hat{\chi}_\pm$ is now contained in the Bogoliubov functions $\mathbf{u}_\lambda(\mathbf{r}, t)$ and $\mathbf{v}_\lambda(\mathbf{r}, t)$, while the mode operators \hat{a}_λ and \hat{a}_λ^\dagger , annihilating or creating an initial particle with energy $|\lambda|$, are time-independent. The initial values of the Bogoliubov functions \mathbf{u}_λ

and \mathbf{v}_λ are (for stable modes, i.e., purely imaginary eigenvalues λ) given by the right eigenfunctions of evolution equations (8), after absorbing the initial phase oscillations of the background Ψ into the linear operators to render the coefficients of (8) time-independent.

3.2. Observables

The calculation of observables is now straight forward. Using Eqs. (7) and (10), density and phase fluctuations (6) can be expressed in terms of Bogoliubov functions, \mathbf{u}_λ and \mathbf{v}_λ , and mode operators \hat{a}_λ , \hat{a}_λ^\dagger

$$\begin{aligned}\delta\hat{\varrho} &= \sqrt{2}(\Re\Psi\hat{\chi}_+ + \Im\Psi\hat{\chi}_-) = \sqrt{2}(\mathbf{u}_\lambda\Re\Psi + \mathbf{v}_\lambda\Im\Psi)\hat{a}_\lambda + \text{H.c.}, \\ \delta\hat{S} &= \frac{1}{\sqrt{2}\varrho}(\Re\Psi\hat{\chi}_- - \Im\Psi\hat{\chi}_+) = \frac{1}{\sqrt{2}\varrho}(\mathbf{v}_\lambda\Re\Psi - \mathbf{u}_\lambda\Im\Psi)\hat{a}_\lambda + \text{H.c.}\end{aligned}\quad (11)$$

Their expectation values $\langle\delta\hat{\varrho}\rangle$ and $\langle\delta\hat{S}\rangle$ must be zero, as they measure only the deviation from the mean, ϱ_0 and S_0 , respectively. But their correlations are usually non-zero. For the density correlations, I obtain

$$\frac{\langle\hat{\varrho}(\mathbf{r})\hat{\varrho}(\mathbf{r}')\rangle}{\langle\varrho(\mathbf{r})\rangle\langle\varrho(\mathbf{r}')\rangle} = 1 + 2\frac{\mathbf{u}_\lambda(\mathbf{r})\Re\Psi(\mathbf{r}) + \mathbf{v}_\lambda(\mathbf{r})\Im\Psi(\mathbf{r})}{|\Psi(\mathbf{r})|^2}\frac{\mathbf{u}_\lambda^*(\mathbf{r}')\Re\Psi(\mathbf{r}') + \mathbf{v}_\lambda^*(\mathbf{r}')\Im\Psi(\mathbf{r}')}{|\Psi(\mathbf{r}')|^2}, \quad (12)$$

and for the phase correlations follows

$$\langle\hat{S}(\mathbf{r})\hat{S}(\mathbf{r}')\rangle = 1 + \frac{1}{2}\frac{\mathbf{v}_\lambda(\mathbf{r})\Re\Psi(\mathbf{r}) - \mathbf{u}_\lambda(\mathbf{r})\Im\Psi(\mathbf{r})}{|\Psi(\mathbf{r})|^2}\frac{\mathbf{v}_\lambda^*(\mathbf{r}')\Re\Psi(\mathbf{r}') - \mathbf{u}_\lambda^*(\mathbf{r}')\Im\Psi(\mathbf{r}')}{|\Psi(\mathbf{r}')|^2}, \quad (13)$$

where I took the expectation values with respect to the vacuum state $|0\rangle$ defined through $\hat{a}_\lambda|0\rangle = 0 \ \forall\hat{a}_\lambda$. Phase correlations could be measured in interference experiments after splitting the condensate [68, 67]. Since this might be difficult to accomplish due to the nontrivial topology of the toroidal condensate, I will focus on density correlations as they can be easily obtained from absorption images [69, 56, 55, 45, 70, 71, 72, 73, 74, 58, 57].

3.3. Time-dependent correlations

Although in Eq. (12), any time arguments were omitted, for a gapped excitation spectrum, the correlation function usually performs temporal oscillations even after the external driving ceased. As will be shown in the following, this residual time-dependence can be linked to phase coherence of squeezed excitations and could thus serve as probe to detect non-adiabatic evolution. For simplicity, I will assume that the condensate settles into a stationary state again after some time t_{end} and thus facilitates the introduction of final quasi-particles. Analogous to Eq. (10), the linear field operators

$$\begin{aligned}\hat{\chi}_+(\mathbf{r}, t) &= \mathbf{f}_\kappa(\mathbf{r}, t)\hat{b}_\kappa + \mathbf{f}_\kappa^*(\mathbf{r}, t)\hat{b}_\kappa^\dagger, \\ \hat{\chi}_-(\mathbf{r}, t) &= \mathbf{g}_\kappa(\mathbf{r}, t)\hat{b}_\kappa + \mathbf{g}_\kappa^*(\mathbf{r}, t)\hat{b}_\kappa^\dagger,\end{aligned}\quad (14)$$

can be expanded in terms of final quasi-particle solutions with creation and annihilation operators \hat{b}_κ^\dagger and \hat{b}_κ . They are related to initial creation and annihilation operators through a Bogoliubov transformation, $\hat{b}_\kappa = \alpha_{\kappa\lambda}\hat{a}_\lambda + \beta_{\kappa\lambda}^*\hat{a}_\lambda^\dagger$, which implicates for the mode functions $\mathbf{u}_\lambda = \mathbf{f}_\kappa\alpha_{\kappa\lambda} + \mathbf{f}_\kappa^*\beta_{\kappa\lambda}$ and $\mathbf{v}_\lambda = \mathbf{g}_\kappa\alpha_{\kappa\lambda} + \mathbf{g}_\kappa^*\beta_{\kappa\lambda}$. The density fluctuations then assume the form $\delta\hat{\varrho} = [h_\kappa(\mathbf{r})e^{-i|\kappa|t}\alpha_{\kappa\lambda} + h_\kappa^*(\mathbf{r})e^{i|\kappa|t}\beta_{\kappa\lambda}]\hat{a}_\lambda + \text{H.c.}$, where the function $h_\kappa(\mathbf{r}) = \mathbf{f}_\kappa(\mathbf{r})\Re\Psi(\mathbf{r}) + \mathbf{g}_\kappa(\mathbf{r})\Im\Psi(\mathbf{r})$ describes the space-dependence

of the final density fluctuation modes but also contains constant factors accounting, e.g., for the initial state, or the vacuum amplitude of the density correlations. Due to the mixing of positive and negative frequency solutions as well as the mixing of different modes during the non-adiabatic evolution, the coefficient of the annihilation operator has many contributions some of which oscillate with positive frequencies $|\kappa|$ and some with negative frequencies $-|\kappa|$. Hence, the equal-time correlator (summation over κ , σ , and λ)

$$\begin{aligned} \langle \delta \hat{\varrho}(\mathbf{r}) \delta \hat{\varrho}(\mathbf{r}') \rangle = & e^{-i(|\kappa|-|\sigma|)t} h_\kappa h_\sigma^* \alpha_{\kappa\lambda} \alpha_{\sigma\lambda}^* + e^{+i(|\kappa|-|\sigma|)t} h_\kappa^* h_\sigma \beta_{\kappa\lambda} \beta_{\sigma\lambda}^* \\ & + e^{-i(|\kappa|+|\sigma|)t} h_\kappa h_\sigma \alpha_{\kappa\lambda} \beta_{\sigma\lambda}^* + e^{+i(|\kappa|+|\sigma|)t} h_\kappa^* h_\sigma^* \beta_{\kappa\lambda} \alpha_{\sigma\lambda}^* \end{aligned} \quad (15)$$

generally comprises many oscillating terms with frequencies $\pm|\lambda| \pm |\kappa|$. Bearing in mind that $h_\kappa(\mathbf{r})$ and $h_\sigma(\mathbf{r}')$ describe the space-dependence of the final density fluctuation eigenmodes, any κ , σ components of (15), e.g., $h_\kappa h_\sigma^* \alpha_{\kappa\lambda} \alpha_{\sigma\lambda}^*$ or $h_\kappa h_\sigma \alpha_{\kappa\lambda} \beta_{\sigma\lambda}^*$, can be obtained by a suitable (spatial and temporal) projection of the (measured) correlation function. After some algebra, the moduli of the Bogoliubov coefficients $|\alpha_{\kappa\lambda}|$ and $|\beta_{\kappa\lambda}|$ and the constant contributions to h_κ follow.

However, since the general calculation is rather tedious, I will discuss the simpler case with diagonal Bogoliubov coefficients $\alpha_{\kappa\lambda}$, $\beta_{\kappa\lambda} \propto \delta_{\kappa\lambda}$ here. Different eigenmodes do not couple and the density correlator (15) assumes the form

$$\langle \delta \hat{\varrho}(\mathbf{r}) \delta \hat{\varrho}(\mathbf{r}') \rangle(t) = h_\kappa(\mathbf{r}) h_\kappa^*(\mathbf{r}') [|\alpha_\kappa|^2 + |\beta_\kappa|^2 + 2 \cos(2|\kappa|t) |\alpha_\kappa \beta_\kappa|], \quad (16)$$

where I omitted the phases of the Bogoliubov coefficients α_κ , β_κ and of the spatial function h_κ in the oscillating term (i.e., absorbed them in the exponentials). The moduli of the Bogoliubov coefficients α_κ and β_κ as well as the modulus of h_κ (i.e., the adiabatic contribution) can be easily calculated from mean value, $h_\kappa h_\kappa^* (|\alpha_\kappa|^2 + |\beta_\kappa|^2)$, and amplitude, $2h_\kappa h_\kappa^* |\alpha_\kappa \beta_\kappa|$, of the temporal oscillations together with the unitarity relation $|\alpha_\kappa|^2 - |\beta_\kappa|^2 = 1$.

Thus a time-resolved measurement of the *equal-time* correlation function (12) provides a direct means of detecting non-adiabatic evolution of the linear quantum fluctuations, i.e., quasi-particles squeezed out of the vacuum can be observed. Any information about the initial state, e.g., whether it is thermal or the vacuum, is only encoded in the prefactor $h_\kappa(\mathbf{r})$ such that a clear discrimination between amplification and initially present particles is possible – a feature the phonon detection scheme proposed in [75] lacks. Note that the correlation function (15) is at *equal* times. Its Fourier transformed should therefore not be confused with the dynamical structure factor $S(k, \omega) = \int d^D k d\omega / (2\pi)^{D/2+1} e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \langle \hat{\varrho}(\mathbf{r}, t) \hat{\varrho}(\mathbf{0}, 0) \rangle$, cf. e.g. [60, 76], which is the Fourier transformed of the density correlations at *different* times.

4. Toroidal condensate

Inspired by recent experiments [18, 19, 20], I will consider a harmonic trap of frequency ω with repulsive Gaussian core of strength V_0 and width σ

$$V(r) = \frac{m\omega^2 r^2}{2} + V_0 \exp \left\{ -\frac{r^2}{\sigma^2} \right\}. \quad (17)$$

For simplicity, I will assume that the condensate is quasi-two-dimensional with radial and azimuthal degrees of freedom. Due to the isotropy of the trap, any mean field motion can only be excited in the radial direction, whereas quantum fluctuations occur at all azimuthal wavenumbers. After integrating out the radial dependence of

the density, $\hat{\varrho}(\phi, t) = \int dr r \hat{\varrho}(\phi, r, t)$, the azimuthal density correlations assume the form

$$\frac{\langle \hat{\varrho}(\phi, t) \hat{\varrho}(\phi', t) \rangle}{\langle \hat{\varrho}(\phi, t) \rangle \langle \hat{\varrho}(\phi', t) \rangle} = 1 + \frac{e^{im(\phi-\phi')}}{\sqrt{2\pi}} \frac{|\delta \varrho_m|^2}{\varrho_0^2} = 1 + \frac{e^{im(\phi-\phi')}}{2\pi} \sum_{\lambda} \frac{|\delta \varrho_m^{\lambda}|^2}{\varrho_0^2}, \quad (18)$$

where

$$\delta \varrho_m^{\lambda} = \int_0^{2\pi} d\phi \int_0^{\infty} dr r \frac{e^{-im\phi}}{\sqrt{\pi}} [\mathbf{u}_{\lambda}(\mathbf{r}, t) \Re \Psi(\mathbf{r}, t) + \mathbf{v}_{\lambda}(\mathbf{r}, t) \Im \Psi(\mathbf{r}, t)] \quad (19)$$

are the contributions of the initial quasi-particle modes λ , see Appendix A for more details.

5. Quench

In the examples, a condensate of 10^5 interacting ^{85}Rb atoms in a quasi-two-dimensional trap with frequency $\omega = 40\pi \text{ Hz}$, cf. Eq. (17), will be considered. The Gaussian beam shall have a size of $\sigma = 175 \mu\text{m}$ and intensity $V_0 = 5000\hbar\omega$ so that the potential minimum of the torus lies at $r_{\min} = 277 \mu\text{m}$. This trap geometry is loosely inspired by those used in Refs. [18, 24], though it should be noted that I use a much higher intensity V_0 of the Gaussian beam and also a lower particle number to facilitate a quadratic approximation of the radial trapping with effective frequency ω_{eff} , see Appendix A for details. Because confinement in the third direction (z) is supposed to be sufficiently tight and enters the calculation only via the effective two-dimensional coupling strength $U_{2\text{D}} = (2\sqrt{2\pi}/m)a_s(t)/a_{\perp}$, I will not give any particular value for ω_{\perp} nor for the initial s -wave scattering length a_s . However, the general features, i.e., the oscillations of the azimuthal density correlations, should be retained for three-dimensional trap geometries, higher particle numbers, or less intense Gaussian cores as well.

The interaction strength is quenched from $U_0 = 0.025\hbar\omega_{\text{eff}}a_{\text{eff}}$ to $U_1 = 0$. The initial chemical potential is $\mu_{\text{in}} = 7.29\omega_{\text{eff}}$, so that the lowest 9 radial trap modes will have significant occupation numbers and the condensate is thus in a quasi-two-dimensional regime. When adiabatically lowering the nonlinear coupling U , the chemical potential would decrease and the population of all radial trap modes except the lowest would decline – the system would become effectively one-dimensional. For rapid changes of U , this, however, is generally not the case any more and (radial) breathing oscillations of the mean field will be excited. Several radial trap modes will have macroscopic occupation. Evidentially, as the final state is not stationary, the definition of a chemical potential is not meaningful any more.

The results presented in the following have been obtained using a basis expansion of the order parameter, Ψ , and the linear fluctuations, $\hat{\chi}_{\pm}$, see Appendix A, and keeping the lowest 15 radial basis functions. The initial state was propagated using a fourth-order Runge-Kutta algorithm with adaptive step width for the coupled evolution equations of mean field (A.2) and fluctuations (A.3).

5.1. Sudden quench

A sudden reduction of the interaction strength U from a finite value to zero is probably the easiest quench dynamics that can be realized in experiments. In Figure 1, the time-dependence of the azimuthal density correlation spectrum is shown for such a

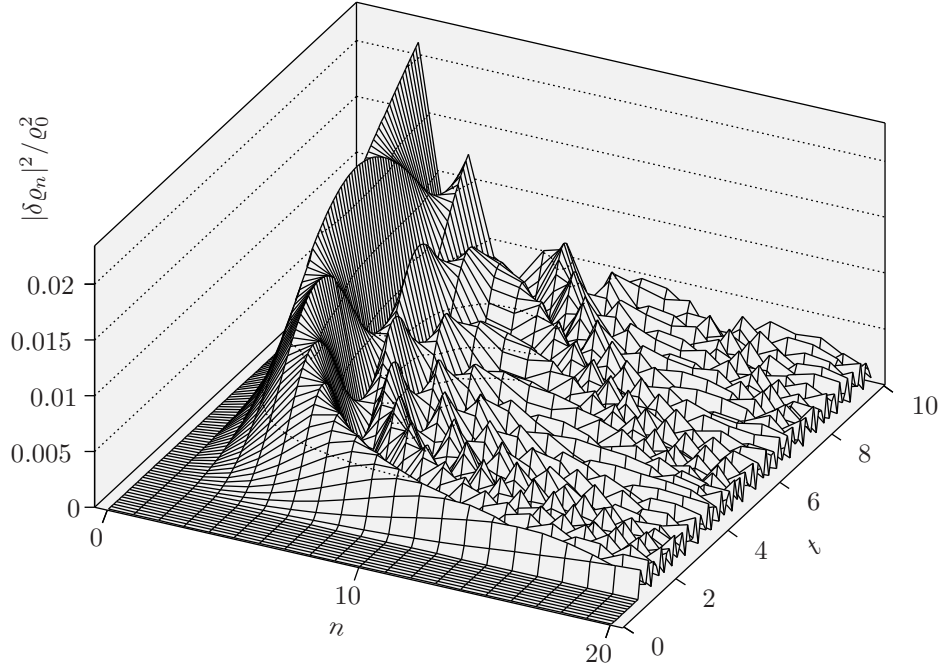


Figure 1. Fourier components $|\delta\rho_n|^2/\rho_0^2$ of the density correlator as a function of azimuthal mode n and time t for a sudden quench of the interaction strength from $U_0 = 0.025 \hbar\omega_{\text{eff}}a_{\text{eff}}$ to $U_1 = 0$ after $t_{\text{sweep}} = 1\text{s}$.

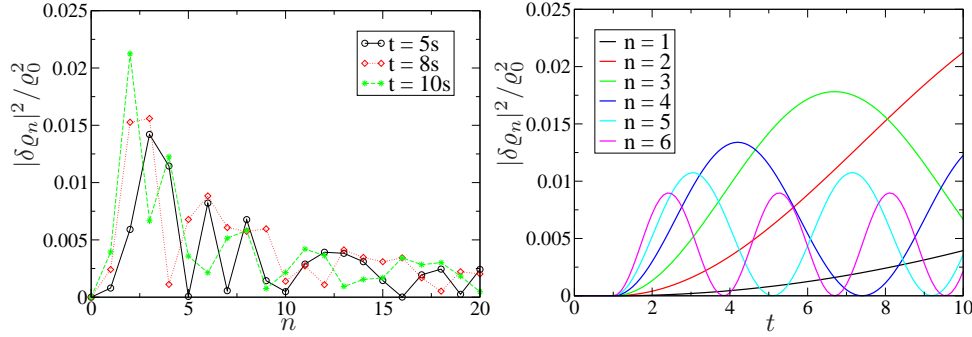


Figure 2. Density-density correlations for sudden quench from $U_0 = 0.025 \hbar\omega_{\text{eff}}a_{\text{eff}}$ to $U_1 = 0$ at $t_{\text{sweep}} = 1\text{s}$. On the left, the spectrum is plotted at different times, whereas the time dependence of the lowest six components $|\delta\rho_1|^2/\rho_0^2$ to $|\delta\rho_6|^2/\rho_0^2$ is shown on the right.

transition from $U_0 = 0.025 \hbar\omega_{\text{eff}}a_{\text{eff}}$ to $U_1 = 0$. Relative density correlations on the percent level can be observed for low azimuthal excitations. However, the $n = 0$ mode is not squeezed as this would correspond to fluctuations in the (conserved) total particle number. The left plot of Figure 2 shows the spectrum at different times,

while the right plot displays the time-dependence of the the lowest azimuthal modes. The correlation spectrum does not converge to a constant value, but rather exhibits temporal oscillations well after the quench even though the interaction strength has been tuned to zero.

These temporal oscillations of correlation functions might seem a bit puzzling at first but are actually a direct consequence of the non-adiabatic evolution during the interaction quench together with a gapped final excitation spectrum, see also Section 3.2 especially Eqs. (15) and (16). The oscillation frequencies observed in the plots are in good agreement with the expected values: After the interactions have been tuned to zero, the azimuthal excitation frequencies are $\omega_n = n^2/2mr_{\min}^2 = n^2 \times 0.306$ Hz above the corresponding mean field in each radial mode. For $n = 4$, $n = 5$, and $n = 6$, this implies period times $T_4 = 6.42$ s, $T_5 = 4.11$ s, and $T_6 = 2.85$ s – the values observed in Figures 1 and 2. For lower excitations, one oscillation period lasts much longer such that no full cycles of $|\delta\varrho_n|^2(t)$ can be observed during the propagation time. Whereas higher excitations undulate very quickly, e.g., $n = 20$ has repetition time $T_{20} = 0.26$ s, making it much harder to resolve in Figure 1 due to finite grid size of the plot. Also, the amplitude of the oscillations declines with n , as it is related to the Bogoliubov $|\beta|$ coefficients, which, for a sudden transition, read

$$|\beta| = \frac{1}{2} \left| \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} - \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \right|. \quad (20)$$

The azimuthal Bogoliubov coefficients, i.e., summed over all radial excitations, can be calculated from minimal and maximal values of the oscillations in Fig. 2 (though it might be difficult to extract these values from the plots). From the numerical data, I have for the mode $n = 4$ as minimum $|\delta\varrho_4|_{\min}^2/\varrho_0^2 = 2.44 \times 10^{-6}$ and as maximum $|\delta\varrho_4|_{\max}^2/\varrho_0^2 = 1.34 \times 10^{-2}$, such that the Bogoliubov coefficient of that azimuthal mode is $|\beta_{n=4}|^2 \approx 18$. This relatively large number can be understood from the huge energy difference before and after the quench: Initially, the condensate is in an interacting regime and the quasi-particles are phonon-like excitations whose frequency is dominated by the radial contribution of order of the effective trap frequency, $\omega_{\text{eff}} = 281$ Hz, cf. Appendix A. After the quench, however, only the small azimuthal $\mathcal{O}(1)$ Hz part remains as the condensate becomes non-interacting.

The oscillations in the azimuthal equal-time density correlations can be interpreted as squeezing of quasi-particle (atom) pairs out of the quantum vacuum. Due to angular momentum conservation, one of the partners must have azimuthal wavenumber $+n$ and one $-n$, i.e., one moves clockwise and one counter-clockwise. If both quasi-particles are at the same place, they will not contribute to the non-local correlations, hence the minima of (16). As they are moving away from each other, their contribution to $\langle \hat{\varrho}(\phi)\hat{\varrho}(\phi') \rangle$ grows until it reaches its maximum when both quasi-particles are farthest apart. A time-resolved measurement of the density-density correlations can be used to trace these coherent pairs and thus infer the amount of squeezing incurred during the transition, e.g., the Bogoliubov coefficients, $|\beta_n|^2$, giving the number of created quasi-particles in each azimuthal mode (summing over the different radial components) [43].

5.2. $\tanh(\gamma t)$ sweep

As second example for the dynamics, I will consider a smooth transition ($\gamma > 0$)

$$U(t) = \frac{U_0 + U_1}{2} - \frac{U_0 - U_1}{2} \tanh[\gamma(t - t_{\text{sweep}})] \quad (21)$$

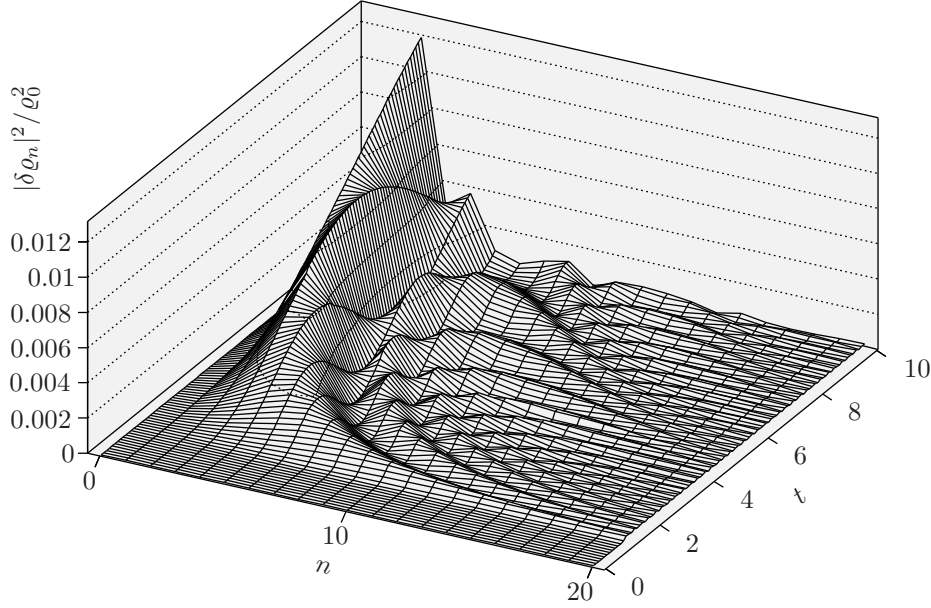


Figure 3. Time-dependence of the correlation spectrum for a smooth tanh-shaped transition between $U_0 = 0.025 \hbar \omega_{\text{eff}} a_{\text{eff}}$ and $U_1 = 0$, cf. Eq. (21). The transition occurs at $t_{\text{sweep}} = 1$ s with sweep rate is equal to the harmonic trap frequency $\gamma = \omega = 40\pi$ Hz.

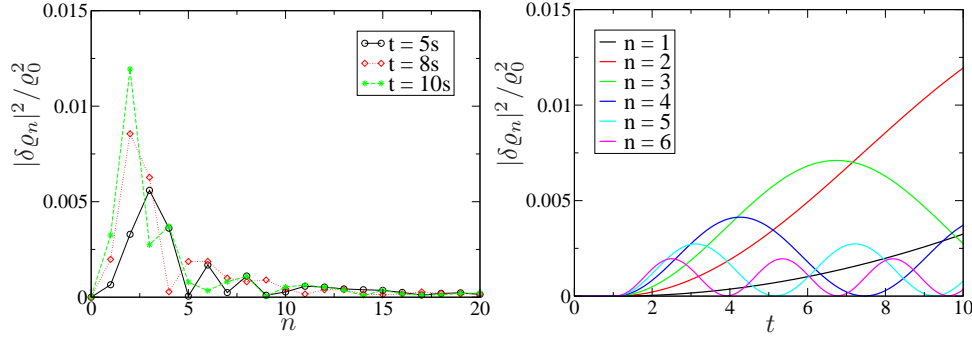


Figure 4. Density-density correlations for a smooth tanh transition from $U_0 = 0.025 \hbar \omega_{\text{eff}} a_{\text{eff}}$ to $U_1 = 0$ with sweep rate $\gamma = \omega = 40\pi$ Hz at $t_{\text{sweep}} = 1$ s, cf. Eq. (21). The left plot shows the spectrum at different times and the right shows the time-dependence of the lowest azimuthal correlation.

from $U_0 = 0.025 \hbar \omega_{\text{eff}} a_{\text{eff}}$ to $U_1 = 0$, i.e., between the same limiting values as before. The sweep rate γ shall be equal to the trap frequency ω , such that only the low-lying excitations with energies of about the same order as the trap frequency will evolve non-adiabatically. The results are plotted in Figures 3 and 4. Again, correlations on the percent level can be observed, though, as expected, smaller than in the previous

example. Temporal oscillations of $|\delta\varrho_n^2|$, characteristic for non-adiabatic evolution, can be observed for low excitations only and subside much more rapidly with increasing n than for a sudden transition. The spectrum $|\delta\varrho_n|^2$ becomes (almost) constant in time for higher-energetic modes $n \approx 15 \dots 20$ already because the excitation energies are smaller than the inverse sweep rate $1/\gamma$ and these higher fluctuations can (at least partially) adapt to changes of the background.

6. Discussion

In summary, I studied the evolution of quasi-two-dimensional Bose-Einstein condensates in isotropic toroidal traps when tuning the (repulsive) interaction strength from a finite value to zero. Since present experiments [18, 19, 20] provide only relatively weak radial confinement, the initial interaction strength was chosen such that the condensate is in an effectively two-dimensional regime with radial and azimuthal degrees of freedom. In the third direction (z), confinement shall be sufficiently tight such that motion is restricted to the trap ground state and can be integrated out.

Due to the isotropy of the trap, the classical mean field must be homogeneous in the azimuthal direction. Hence, any background motion must occur radially. The linear quantum fluctuations, on the other hand, are only subject to (angular) momentum conservation and might thus have non-trivial azimuthal dependence, which can be treated through a Fourier transformation. For each Fourier mode, two coupled one-dimensional field equations (in the radial direction) with gap $n^2/2mr_{\min}^2$ follow, which can be easily solved numerically. By integrating out the radial dependence and considering only the (relative) azimuthal correlations, a clear signature of the quantum fluctuations can be provided.

The radial dependence of mean field and linear quantum fluctuations was expanded in terms of harmonic oscillator functions. For the initial finite interaction strength, the condensate (mean field) is radially distributed over several modes. When slowly (adiabatically) reducing the interaction strength U , more and more atoms would gather in the trap ground state, while the occupation of all higher modes would diminish until the condensate becomes effectively one dimensional. For rapid variations of U , however, the condensate might not be able to follow these changes of the interaction strength and will therefore not stay in its ground state. A fraction of the atoms would remain in higher trap states and the mean field would split up into several parts, whose phases oscillate with different frequencies – the condensate would undergo classical oscillations in the radial direction. It should be noted that, even though the condensate performs breathing motion and is thus split into different parts evolving with vastly different frequencies, the mean field is still coherent, i.e., the phases of the classical excitations and the non-excited bulk are still uniquely related. Coupling to an environment or the back-reaction of quantum fluctuations might, however, lead to loss of phase coherence so that the condensate becomes fragmented into several mutually incoherent parts [78, 79].

For non-vanishing interaction strength after the quench, $U \neq 0$, and if the mean field undergoes radial breathing, the background-dependent terms in the linear field equations, $U|\Psi(r, t)|^2$ and $U\Psi^2(r, t)$, would become periodic in time and could amplify quantum fluctuations through resonance. Therefore, I considered only the case, where U is tuned to zero and the theory becomes non-interacting, i.e., the linear excitations evolve independently of the condensate background. In view of the azimuthal homogeneity of the mean field, azimuthal density correlations – the

radial dependence has been integrated out – represent a suitable observable for the quantum fluctuations independent of any background motion. Two different time-dependences of the interaction strength were considered: firstly, a sudden change from a finite U_0 to zero and, secondly, a smooth tanh-shaped transition between the same values of the interaction strength. In both cases, squeezing of quasi-particles could be observed, which manifests as temporal oscillations in the azimuthal correlation functions. This can be understood in a simple picture: due to angular momentum conservation, quasi-particles are always excited as pairs with azimuthal wavenumbers $\pm n$. With one quasi-particle (atom) circling the torus clockwise and the other counter-clockwise, the oscillations in the correlation function can then be understood through co-incidence of the pair. For a sudden transition, all modes evolve non-adiabatically, whereas a smooth tanh shape of the interaction strength violates adiabaticity only for low excitations, while higher modes stay closer to their ground state. In both cases, relative (vacuum) density-density correlations on the percent level could be observed.

Temporal oscillations in the equal-time correlation function after the quench are closely related to finite quasi-particle frequencies and non-adiabatic evolution of the fluctuations – initial particle solutions oscillating with positive frequency before the quench usually comprise positive and negative frequency parts afterwards, where the amount of mixing can be quantified through Bogoliubov coefficients α and β . For non-zero β , i.e., non-adiabatic evolution, the density correlator will assume oscillations with twice the final quasi-particle frequency. From mean value and amplitude of the oscillations, the Bogoliubov coefficients as well as the adiabatic correlations can be obtained. Thus, a *time-resolved* measurement of the *equal-time* density-density correlation function permits the determination of the Bogoliubov coefficients $|\beta|^2$ as signature of quasi-particle squeezing.

In contrast to methods aiming at the detection of single phonons, e.g., [75], the temporal oscillations of the density correlation are as signature of squeezing relatively robust against changes of the initial quantum state. For instance, a thermal occupation number would merely appear as prefactor to the correlations of a mode, whereas the Bogoliubov coefficients $|\beta|$ as measure for the amplification of the initial fluctuations (and quasi-particles) follow from the relative amplitude compared to the mean of the oscillations, which is unaffected by such a prefactor. If the excitation frequencies would go to zero, which happens in expanding Bose-Einstein condensates after the trap has been turned off [46, 55, 56, 58, 57] or during cosmic inflation [42, 44], the density correlations would converge and a time-resolved measurement would yield no additional information. In particular, it would not be possible to distinguish between contributions stemming from initial particles and those due to the amplification of fluctuations, i.e., squeezing.

A time-dependent measurement of the *equal-time* density correlations is also fundamentally different from obtaining the dynamical structure factor [60, 76], which is the temporal Fourier transformed of the density correlator at *different* times. To exemplify these differences, let me consider a simple one-mode model $\hat{\varrho} = f(t)\hat{a} + f^*(t)\hat{a}^\dagger$. Assuming that the system is in squeezed state, $f(t) = \alpha e^{-i\omega t} + \beta^* e^{i\omega t}$, the propagator reads $\langle \hat{\varrho}(t)\hat{\varrho}(t') \rangle = f(t)f^*(t') = (|\alpha|^2 + \alpha\beta e^{-2i\omega t'})e^{-i\omega(t-t')} + (|\beta|^2 + \alpha^*\beta^* e^{2i\omega t'})e^{i\omega(t-t')}$ and the dynamical structure factor has two peaks at $\pm\omega$ with time-dependent coefficients. It is, in principle, possible to obtain the Bogoliubov coefficients and thus the amount of squeezing from the time-dependence of these peaks as well, though a time-dependent measurement where *both* times t and t' are varied would be necessary.

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Appendix A. Basis expansion

Appendix A.1. General basis

Method of choice for the numerical propagation of mean field Ψ as well as quantum fluctuations $\hat{\chi}_{\pm}$ is a basis expansion. To remain general, I will formally consider non-orthogonal bases in the following, i.e., tensor notation will be adopted and I will distinguish between covariant $\{f_{\alpha}(\mathbf{r})\}$ and contravariant $\{f^{\alpha}(\mathbf{r})\}$ bases. They are normalized $\langle f^{\alpha}, f_{\beta} \rangle_{L^2} = \delta^{\alpha}_{\beta}$ with L^2 inner product $\langle f, g \rangle_{L^2} = \int d^D r f^*(\mathbf{r})g(\mathbf{r})$. Non-orthogonal basis might occur, e.g., when using the eigenfunctions of a non-symmetric operator as basis, cf. also D [65], or when picking a particularly suitable numerical basis.

The components of the order parameter and the linear fluctuations

$$\Psi(\mathbf{r}, t) = f_{\alpha}(\mathbf{r})\Psi^{\alpha}(t), \quad \hat{\chi}_{\pm}(\mathbf{r}, t) = f_{\alpha}(\mathbf{r})\hat{\chi}_{\pm}^{\alpha}(t) \quad (\text{A.1})$$

follow from their projection onto the covariant basis $\Psi^{\alpha} = \langle f^{\alpha}, \Psi \rangle_{L^2}$ and $\hat{\chi}_{\pm}^{\alpha} = \langle f^{\alpha}, \hat{\chi}_{\pm} \rangle_{L^2}$. The Gross-Pitaevskii equation (3) reads in basis expansion

$$i\partial_t \Psi^{\alpha} = \mathcal{K}^{\alpha}_{\beta} \Psi^{\beta} + U \mathcal{M}^{\alpha}_{\beta\gamma\delta} (\Psi^{\beta})^* \Psi^{\gamma} \Psi^{\delta}, \quad (\text{A.2})$$

where the entire space-dependence (and possibly the time-dependence of the external confinement V) is integrated out and thus contained in the coupling tensors $\mathcal{K}^{\alpha}_{\beta} = \int d^D r (f^{\alpha})^* (-\nabla^2/2m + V) f_{\beta}$ and $\mathcal{M}^{\alpha}_{\beta\gamma\delta} = \int d^D r (f^{\alpha})^* f_{\beta}^* f_{\gamma} f_{\delta}$. For the linear fluctuations, I obtain

$$i\partial_t \begin{pmatrix} \hat{\chi}_{+}^{\alpha} \\ \hat{\chi}_{-}^{\alpha} \end{pmatrix} = \begin{pmatrix} \mathcal{C}^{\alpha}_{\beta} & \mathcal{A}^{\alpha}_{\beta} \\ -\mathcal{B}^{\alpha}_{\beta} & -\mathcal{C}^{\alpha}_{\beta} \end{pmatrix} \cdot \begin{pmatrix} \hat{\chi}_{+}^{\beta} \\ \hat{\chi}_{-}^{\beta} \end{pmatrix}. \quad (\text{A.3})$$

The coupling tensors $\mathcal{A}^{\alpha}_{\beta} = \int d^D r (f^{\alpha})^* \mathcal{A} f_{\beta}$, $\mathcal{B}^{\alpha}_{\beta} = \int d^D r (f^{\alpha})^* \mathcal{B} f_{\beta}$, and $\mathcal{C}^{\alpha}_{\beta} = \int d^D r (f^{\alpha})^* \mathcal{C} f_{\beta}$ follow by taking the matrix elements of operators (9) with respect to the basis $\{f_{\alpha}\}$ and its dual $\{f^{\alpha}\}$. Note that, due to the non-linearity of the original field equation (1) and thus the appearance of the mean field Ψ in the linear evolution equations (8) for $\hat{\chi}_{\pm}$, the matrix coefficients in (A.3) usually contain integrals of four basis functions similar to $\mathcal{M}^{\alpha}_{\beta\gamma\delta}$ and summation over the components of the order parameter Ψ .

From Eqs. (10) and (A.1), I can infer the basis expansion of the Bogoliubov functions \mathbf{u}_{λ} and \mathbf{v}_{λ}

$$\mathbf{u}_{\lambda}(\mathbf{r}, t) = f_{\alpha}(\mathbf{r})\mathbf{u}_{\lambda}^{\alpha}(t), \quad \mathbf{v}_{\lambda}(\mathbf{r}, t) = f_{\alpha}(\mathbf{r})\mathbf{v}_{\lambda}^{\alpha}(t). \quad (\text{A.4})$$

so that the linear field operators $\hat{\chi}_{\pm}$ can be described by time-independent operators $\hat{a}_{\lambda}^{\dagger}$, \hat{a}_{λ} creating or annihilating an initial quasi-particle with energy $|\lambda|$ and time-dependent coefficients $\mathbf{u}_{\lambda}^{\alpha}(t)$ and $\mathbf{v}_{\lambda}^{\alpha}(t)$. The initial values of the Bogoliubov functions are given by the (up and down) components of the right eigenvectors of the coupling matrix appearing in (A.3), and can be propagated using Eq. (A.3). As the evolution equation is first order in time, positive and negative frequency solutions appear separately in the spectrum and one needs to be careful not to double count the

pair, i.e., (for stable modes) eigenvalues with negative imaginary part correspond to annihilation operators \hat{a}_λ and those with positive imaginary part yield creators \hat{a}_λ^\dagger . Inserting expansions (A.1) and (A.4) into (12), observables such as the density correlations can be calculated. One should, however, bear in mind that the basis functions f_α are usually not real and thus complex conjugation of the coefficients u_λ^α etc. does not yield the coefficients of $(u_\lambda^\alpha)^*$ etc. with respect to the basis $\{f_\alpha\}$ but rather those with respect to $\{f_\alpha^*\}$.

Appendix A.2. Toroidal condensates

Toroidal Bose-Einstein condensates can be created using potential (17), which has its minimum at $r_{\min}^2 = \sigma^2 \ln(2V_0/m\omega^2\sigma^2) > 0$ for $2V_0 > m\omega^2\sigma^2$. In order to simplify the analysis a bit, this potential can be expanded to second order about this minimum

$$V(r) = V(r_{\min}) + \frac{m\omega_{\text{eff}}^2}{2}(r - r_{\min})^2 + \mathcal{O}[(r - r_{\min})^3], \quad (\text{A.5})$$

and an effectively harmonic potential with frequency $\omega_{\text{eff}}^2 = 2\omega^2 \ln(2V_0/m\omega^2\sigma^2) = 2\omega^2 r_{\min}^2/\sigma^2$ can be obtained. The first term, $V(r_{\min})$, yields only a constant energy shift and will be omitted in the following. This expansion (A.5) of the potential can be used if the radial extend Δr of the condensate is much smaller than the torus radius r_{\min} , which can be achieved by demanding that the harmonic oscillator length $a_{\text{eff}} = 1/\sqrt{m\omega_{\text{eff}}}$ be much smaller than r_{\min} and only the lowest few radial modes being occupied, i.e., a chemical potential of order ω_{eff} .

In view of the symmetries of the trap – the potential depends only on $r = |\mathbf{r}|$ and is independent of the azimuthal angle ϕ – it is advantageous to change to planar polar coordinates, $\mathbf{r} = r\mathbf{e}_r(\phi)$, and to introduce bases for radial and azimuthal dependence. Harmonic oscillator functions $h_\alpha(r - r_{\min})$ centered at r_{\min} seem to be an obvious choice for the radial basis. But one should, however, bear in mind that polar coordinates yield an additional factor r as measure in the inner product $\int d^2r h_\alpha^* h_\beta = \int r dr d\phi h_\alpha^* h_\beta$. It is therefore more convenient to absorb this factor and to use slightly different functions

$$f_\alpha(r - r_{\min}) = \frac{h_\alpha(r - r_{\min})}{\sqrt{r}}, \quad (\text{A.6})$$

which are approximately orthonormal $\int_0^\infty dr r f_\alpha(r - r_{\min}) f_\beta(r - r_{\min}) = \delta_{\alpha\beta}$. Corrections to this orthonormality are exponentially suppressed $\propto e^{-(\alpha+\beta)a_0/r_{\min}}$. [Of course, this suppression does not hold for higher modes $\alpha \gtrsim r_{\min}/a_0$, but the occupation numbers in these modes are very small because the condensate is radially localized near r_{\min} . Also, expansion (A.5) of the radial potential (17) to second order would break down first.]

In view of the isotropy of the potential (17), the azimuthal dependence of mean field Ψ and linear fluctuations $\hat{\chi}_\pm$ is most conveniently described in terms of plane waves

$$g_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m \in \mathbb{Z} \quad (\text{A.7})$$

obeying periodic boundary conditions, $g_m(\phi) = g_m(\phi + 2\pi)$, and normalization $\int_0^{2\pi} d\phi g_m^*(\phi) g_n(\phi) = \delta_{nm}$. Due to isotropy of the trap (17), the classical mean field, $\Psi(r, \phi, t) = \frac{1}{\sqrt{2\pi}} f_\alpha(r) \Psi^\alpha(t)$, must be independent of the angle ϕ and the azimuthal label on $\Psi^{m=0,\alpha}$ can be omitted. The Bogoliubov functions $u_\lambda(\mathbf{r}, t) =$

$g_m(\phi)f_\alpha(r)\mathbf{u}_\lambda^{m\alpha}(t)$ and $\mathbf{v}_\lambda(\mathbf{r},t) = g_m(\phi)f_\alpha(r)\mathbf{v}_\lambda^{m\alpha}(t)$, however, depend on ϕ and can be propagated through Eq. (A.3). Different azimuthal modes decouple (except, of course, $\pm n$), because the background does not depend on ϕ so that for each angular basis function, $g_n(\phi) \propto e^{in\phi}$, a set of coupled evolution equations follows.

Since I consider fluctuations above a dynamical background, I must choose observables affected as little as possible by any mean field motion. For any changes of the interaction strength, radial breathing oscillations of the order parameter Ψ are usually excited, whereas the background stays at rest in azimuthal direction. I will therefore pick azimuthal density correlations in the following, where the radial dependence has been integrated out

$$\hat{\varrho}(\phi) = \int dr r \hat{\varrho} = \frac{N}{2\pi} + g_m(\phi)\delta\varrho_\lambda^m \hat{a}_\lambda + g_m^*(\phi)(\delta\varrho_\lambda^m)^* \hat{a}_\lambda^\dagger \quad (\text{A.8})$$

The measure r has been included for convenience. The mean azimuthal density, $N/2\pi$, is constant due to isotropy of potential (17), while the coefficients $\delta\varrho_\lambda^m = (1/\sqrt{\pi}) \sum_\alpha (\mathbf{u}_\lambda^{m\alpha} \Re \Psi^\alpha + \mathbf{v}_\lambda^{m\alpha} \Im \Psi^\alpha)$ of the linear density fluctuations give the nontrivial contributions of the initial eigenmode λ to the density fluctuations with azimuthal wavenumber m , so that the normalized equal-time density-density correlations follow

$$\frac{\langle \hat{\varrho}(\phi) \hat{\varrho}(\phi') \rangle}{\langle \hat{\varrho}(\phi) \rangle \langle \hat{\varrho}(\phi') \rangle} = 1 + \frac{e^{im(\phi-\phi')}}{\sqrt{2\pi}} \frac{|\delta\varrho_m|^2}{\varrho_0^2} \quad (\text{A.9})$$

with $|\delta\varrho_m|^2 = \sum_\lambda |\delta\varrho_\lambda^m|^2 / \sqrt{2\pi}$.

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- [65] Note that the operator \underline{D} is not symmetric with respect to the usual $L^2 \times L^2$ inner product. Although it is possible to introduce a Klein-Gordon type product with respect to which \underline{D} is self-adjoint [30], this comes at the cost of losing positive definiteness (of the inner product). I will use the usual $L^2 \times L^2$ scalar product throughout this article, i.e., discuss the eigenvalue problem of a non-symmetric operator \underline{D} . For real non-symmetric operators, right

and left eigenfunctions, i.e., the eigenfunctions of $\underline{\mathcal{D}}$ and its adjoint $\underline{\mathcal{D}}^\dagger$, are generally distinct but still have the same spectrum with eigenvalue pairs $\pm\lambda$ and $\pm\lambda^*$. Usually, right (left) eigenfunctions are not mutually orthogonal, but an orthogonality relation between left and right eigenfunction holds, i.e., any left eigenfunction is orthogonal to all right eigenfunctions with different eigenvalues. Also, the set of eigenfunctions does not always span the entire Hilbert space ($L^2 \times L^2$), though this is usually the case. It should be noted that in certain situations, e.g., for a homogeneous phase S_0 of the order parameter Ψ [66], the eigenproblem of $\underline{\mathcal{D}}$ can be mapped onto that of a symmetric operator so that the eigenfunctions of $\underline{\mathcal{D}}$ can be shown to be complete.

- [66] For instance, if the background phase is homogeneous, a (possibly time-dependent) phase transformation can render \mathcal{C} zero. The operator $\underline{\mathcal{D}}^2$ will then become diagonal with elements $-\mathcal{A}\mathcal{B}$ and $-\mathcal{B}\mathcal{A}$. Since \mathcal{A} and \mathcal{B} are both self-adjoint, it is possible to define $\sqrt{\mathcal{A}}$ and $\sqrt{\mathcal{B}}$ through the spectral theorem. The eigenvalue problem of $\underline{\mathcal{D}}^2$ can then be rewritten as

$$\begin{pmatrix} -\sqrt{\mathcal{B}}\mathcal{A}\sqrt{\mathcal{B}} & 0 \\ 0 & -\sqrt{\mathcal{A}}\mathcal{B}\sqrt{\mathcal{A}} \end{pmatrix} \begin{pmatrix} \sqrt{\mathcal{B}}u_\lambda \\ \sqrt{\mathcal{A}}v_\lambda \end{pmatrix} = \lambda \begin{pmatrix} \sqrt{\mathcal{B}}u_\lambda \\ \sqrt{\mathcal{A}}v_\lambda \end{pmatrix},$$

which is that of an symmetric operator.

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